

## MONSTROUS STRING-STRING DUALITY

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We analyze the general class of supersymmetry preserving orbifolds of strong/weak Type IIA/heterotic dual pairs in six dimensions and below. A unified treatment is given by considering compactification to two spacetime dimensions and constructing orbifolds by subgroups of the Fischer-Greiss monster, utilizing the moonshine results of Conway and Norton. Duality requires nontrivial Ramond-Ramond fluxes on the Type IIA side which are localized at the fixed points. Further orbifolding by  $(-1)^{F_L}$  gives examples of new four dimensional N=2 Type IIA vacua which are not conformal field theory backgrounds.

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## 1. Introduction

Of the many duality relationships linking compactifications of the mysterious eleven dimensional **M** theory perhaps the best understood are the conjectured string-string dualities, in which a strongly coupled string theory describes another weakly coupled string theory [1–11]. The evidence for a strong/weak duality relating the heterotic string compactified on  $T^4$  and the Type IIA string compactified on  $K3$  includes the matching of their moduli spaces [12–14], a transformation of their respective low-energy effective field theory limits [1,2], and an explicit construction of a nonsingular heterotic string soliton in the Type IIA theory [4,5]. The necessary nonabelian gauge bosons appear as collective coordinates of the soliton in the Ramond-Ramond sector.

In this paper we provide a general analysis of supersymmetry preserving orbifolds of strong/weak Type IIA/heterotic dual pairs in six dimensions and below. The heterotic soliton construction of Harvey and Strominger, and of Sen, provides a map between target space duality symmetries of the IIA theory and corresponding transformations on the world-sheet fields of the heterotic compactification. Provided such a symmetry is *freely acting*, one expects the theories obtained by orbifolding to provide additional strong/weak dual pairs [7,15,16,17,18,19,20–21]. Thus, from this duality alone one can derive a remarkable web of strong/weak dualities in dimensions six and below.

In previous work [18] we obtained dual pairs by orbifolding the IIA theory on  $K3$  by an abelian symplectic automorphism [22], i.e., an automorphism which leaves fixed the holomorphic two-form on  $K3$ . Such automorphisms always have *fixed points* on  $K3$ . To ensure that the symmetry is freely acting, the automorphism is accompanied by a shift corresponding to a Wilson line for the ten-dimensional Ramond-Ramond gauge field of the Type IIA theory [16]. The duality map determines a corresponding automorphism plus shift of the Narain lattice on the heterotic side. It is interesting that abelian symplectic automorphisms of the classical geometry automatically satisfy level matching [18,19], so that the accompanying shift can be chosen to be geometrical. This will not be true more generally.

The symplectic automorphisms of the classical geometry of  $K3$ , including nonabelian cases, have been classified by Mukai [23]. He notes that they are in one-to-one correspondence with the purely left-moving symmetries of the  $(19,3)$  Lorentzian self-dual lattice formed by  $H^2(K3, \mathbb{Z})$ , fitting into special subgroups of the finite sporadic group  $M_{23}$ . An obvious extension is to consider automorphisms of the quantum geometry of the IIA theory on  $K3$ , and its further toroidal compactifications.

A new feature of the orbifolds we consider is the possibility of Wilson line backgrounds for the 24 Ramond-Ramond  $U(1)$  gauge fields of the IIA theory on  $K3$ . Gauge transformations in these  $U(1)$  gauge fields are mapped under duality to general shifts in the  $\Gamma^{(20,4)}$  Narain lattice of the heterotic string on  $T^4$ . By considering orbifolds with respect to these more general symmetries, combined with the action of a symplectic automorphism, we obtain Type IIA theories with RR fluxes localized at the fixed points of  $K3$ . We find that there is considerable freedom in the choice of RR fluxes. Each choice is mapped under duality to a specific shift vector on the heterotic side. The different possibilities for the shift vector give otherwise identical low energy limits, and are therefore associated with T-duality symmetries. It should be noted that from the viewpoint of the Type IIA com-

pactification, these are T-duality symmetries of the Ramond-Ramond sector and do not originate in conformal field theory!

We will show how all these orbifolds fit into a unified structure by considering compactifications down to two dimensions. We make the natural conjecture that the supersymmetry preserving symmetries of the Type II theory on  $K3 \times T^4$  correspond to the purely left-moving symmetries of a  $(24, 8)$  self-dual Lorentzian lattice. We make an additional restriction that the lattice takes the form  $\Gamma^{(24,0)} \oplus \Gamma^{(0,8)}$ . Such a point in the moduli space can always be reached on the Type II side by turning on suitable  $B$  field deformations. The 24-dimensional Euclidean lattices have been classified and are the Leech lattice and its cousins the 23 other Niemeier lattices [24]. The other Niemeier lattices lead to heterotic theories with enhanced gauge symmetry, which are expected to be dual to Type II theories on singular  $K3 \times T^4$  manifolds. To avoid such subtleties we will focus mostly on orbifolds of the Leech lattice.

The automorphism group of the Leech lattice is one of the simple finite groups  $\cdot 0$ , the Conway group. This group is closely related to the monster group  $\mathbb{M}$ , the largest of the sporadic groups. The group  $\mathbb{M}$  is the automorphism group of the monster module conformal field theory  $V^\natural$  obtained as a  $\mathbb{Z}_2$  orbifold of the Leech lattice theory, where the  $\mathbb{Z}_2$  acts by changing the sign of the coordinates on the lattice. This  $\mathbb{Z}_2$  orbifold of the Leech lattice theory contains no massless states in its twisted sectors – therefore we expect there will be a dual IIA theory, obtained via a freely acting orbifold of the quantum geometry. One may then consider further orbifolds of the theories based on  $V^\natural$ , or simply the Leech lattice, yielding a large class of candidate dual pairs with maximal supersymmetry. Note that, in contrast to the approach of [6,25], the soliton string construction gives an unambiguous method to obtain reliable dual pairs as orbifolds by freely acting symmetries.

Having established a duality map in two spacetime dimensions we can lift the action of the supersymmetry preserving automorphism into higher dimensions by decompactifying any invariant toroidal coordinates. In particular, all of the six dimensional classical orbifolds of [16,18] can be recovered by this procedure.<sup>1</sup> As mentioned above, all of the classical  $K3$  automorphisms fit into subgroups of  $M_{23}$ , leaving invariant an  $\Gamma^{(1,1)}$  component of the  $(20, 4)$  lattice. This is not true for the quantum automorphisms— we find symplectic automorphisms that lie in  $M_{24}$ , but not in  $M_{23}$ . We also find quantum symplectic symmetries of  $K3 \times T^4$  that require a further extension to the Conway group.

The plan of the paper is as follows. In section 2 we clarify the duality dictionary between symmetries of the toroidal left-moving conformal field theory on the heterotic side and symplectic symmetries of  $K3$  on the Type II side. In section 3 we consider dual Type IIA/heterotic compactifications to two dimensions. Using the monstrous moonshine results of Conway and Norton, we present the general construction for cyclic orbifolds of the Leech lattice and the Moonshine module, giving an implicit definition of Type IIA/heterotic duals which cannot be obtained by a consideration of classical symmetries of  $K3$  alone. In Section 4, we explain how our construction can be easily modified to

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<sup>1</sup> We are assuming that the  $B$  field may be turned off in a way that preserves the symmetry. To prove this statement requires a more detailed knowledge of the moduli space of quantum  $K3$  surfaces than is presently available.

obtain  $N=2$  Type IIA/heterotic dual pairs. We conclude with some comments and a brief discussion of our results.

## 2. The String-String Duality Dictionary

We begin by considering the strong/weak coupling duality between the heterotic string on  $T^4$  and the Type IIA string on  $K3$ . The heterotic compactification is described by a Narain lattice  $\Lambda = \Gamma^{(20,4)}$ , i.e. an even self-dual Lorentzian lattice. Using the construction of the heterotic string as a soliton of the IIA theory [4,5], duality maps this lattice to a point in the moduli space of the IIA string on  $K3$ . Likewise, symmetries of the heterotic CFT will be mapped to symmetries of the IIA theory. To see this explicitly, we need the expressions for the zero modes of the soliton, which give rise to the worldsheet fields of the heterotic string. Zero modes coming from the Ramond-Ramond (RR) three-form potential are

$$C = \frac{\alpha'}{2\pi} X^I(\sigma) U_I(y) \wedge de^{2\phi_0 - 2\phi(x)} , \quad (2.1)$$

in the notation of [4], with  $\phi(x)$  the background dilaton field. The  $U_I(y)$  are the harmonic two-forms of  $K3$ , with  $I = 1, \dots, 22$ . The worldsheet fields satisfy the chiral constraint

$$*U_I = U_J H_I^J , \quad (2.2)$$

where  $H$  has signature  $(19,3)$ , yielding 19 left-moving and 3 right-moving worldsheet coordinates. An additional zero mode comes from

$$A = \frac{1}{2\pi} X^0(\sigma) de^{2\phi_0 - 2\phi(x)} , \quad C = -\frac{1}{\pi} X^0(\sigma) e^{2\phi_0 - 2\phi(x)} H , \quad (2.3)$$

which gives a single (left,right)-moving bosonic zero-mode. Here  $A$  is the RR one-form potential of the IIA theory in ten dimensions, and  $H$  is the three-form field strength. There are further bosonic zero modes arising from transverse motion in six-dimensional Minkowski space, together with fermionic zero modes, but these will not be relevant in the following discussion.

Let us see how the duality dictionary works for the Type II dual of the simplest of the CHL models [26,16]. On the heterotic side, the model is constructed as an orbifold by a  $\mathbb{Z}_2$  symmetry which exchanges the two  $E_8$  components of the lattice and acts with a half period shift on one of the circles of the torus  $T^4$ . On the Type II side, the exchange of the  $E_8$ 's corresponds to a certain symplectic automorphism of  $K3$ .<sup>2</sup> The shift is identified with a  $\mathbb{Z}_2$  gauge transformation of the  $U(1)$  Ramond-Ramond (RR) gauge field  $A$ . The resulting orbifold on the Type II side is not a conventional superconformal field theory. A nontrivial Wilson line background of the RR gauge field is present, with fluxes localized at the fixed points of the  $K3$ .

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<sup>2</sup> Note that it is possible to move away from this point of enhanced gauge symmetry, preserving the  $\mathbb{Z}_2$  symmetry, so that  $E_8 \times E_8$  is broken down to  $U(1)^{16}$ . Thus one need not consider singular  $K3$  surfaces.

There are a total of 24 RR  $U(1)$  gauge fields present in the IIA theory on  $K3$ . One comes from the 10d RR  $U(1)$  gauge field as mentioned above, one arises from the dual of the RR four-form field strength, and 22 arise from the RR three-form potential integrated over a nontrivial two-cycle of  $K3$ , i.e. we may write

$$C = U_I \wedge A^I . \quad (2.4)$$

It is natural therefore to consider orbifolds which turn on Wilson line backgrounds in these other RR gauge fields. It is clear from (2.1),(2.3) that these nontrivial gauge transformations are equivalent to shifts in the  $X^I$  and  $X^0$ . On the heterotic side, the inclusion of these Wilson lines will correspond to orbifolds involving general shifts of the  $\Gamma^{(20,4)}$  lattice.

In the following section, we will combine these general shifts with supersymmetry preserving lattice automorphisms to construct a large class of maximally supersymmetric dual pairs. The classical symplectic automorphisms of  $K3$  will map to lattice automorphisms [23]. However the complete set of lattice automorphisms are expected to map onto more general quantum symmetries of the IIA string on  $K3$  which are difficult to describe explicitly. However, from the heterotic point of view, these quantum symmetries are precisely on the same footing as the classical symmetries. This suggests that the Type II orbifolds by these quantum symmetries will be implicitly defined by their heterotic counterparts.

So far, we have discussed the mapping of symmetries under duality from the heterotic string on  $T^4$  to the IIA string on  $K3$ . Clearly the same kind of arguments will go through when one considers further toroidal compactification on an additional  $T^4$ , as described in the following section.

### 3. Supersymmetry Preserving Orbifold Dual Pairs

We will begin by considering compactifications to two spacetime dimensions described by  $(24, 8)$  dimensional Lorentzian self-dual lattices<sup>3</sup>

$$\Gamma^{(24,8)} = \Lambda \oplus \Gamma^8 , \quad (3.1)$$

where  $\Gamma^8$  is the  $E_8$  lattice. For the left moving lattice, we take the rank 24 self-dual lattice  $\Lambda_{A_1^{24}}$ . This lattice consists of the  $(SU(2))^{24}$  root lattice plus the conjugacy classes (doublets) obtained from the following generators [24],

$$(1, [0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1]) , \quad (3.2)$$

where the square brackets denote cyclic permutations. The Leech lattice,  $\Lambda_{Leech}$ , is easily generated from  $\Lambda_{A_1^{24}}$  as an orbifold by the purely left moving  $\mathbb{Z}_2$  shift

$$h : X \rightarrow X + 2\pi \cdot \frac{1}{4} \sum_{i=1}^{24} a_i , \quad (3.3)$$

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<sup>3</sup> This construction was suggested to us by J. Harvey.

where the  $a_i$  are the 24 positive roots of  $(SU(2))^{24}$  [27]. Since the Leech lattice has no vectors of length squared two the only massless states in this conformal field theory are the 24  $U(1)$  bosonic states  $|\alpha_{-1}^i\rangle$ .

The automorphism group of this conformal field theory is known as the Conway group, which is a close relative of the monster group  $\mathbb{M}$ , the largest of the sporadic groups. The group  $\mathbb{M}$  is the automorphism group of the monster module conformal field theory, obtained by the  $\mathbb{Z}_2$  involution

$$g : X \rightarrow -X , \quad (3.4)$$

for all 24 coordinates on the Leech lattice. The monster module conformal field theory has no massless states in its twisted sectors. Thus, for each supersymmetry preserving automorphism of the monster module we expect a dual IIA theory obtained via a freely acting orbifold of the quantum geometry.

We will consider supersymmetry preserving automorphisms of the two, closely related, conformal field theories based on these lattices. Many of our results extend to more general lattices with signature  $(24, 8)$ . Our restriction is helpful for technical reasons. We will be interested in symmetries which act purely on the left-moving coordinates of the lattice. Orbifolds will be constructed by combining the action of these symmetries with shifts with right-moving components, ensuring no massless states appear in the twisted sectors. For a subclass of these theories the Type II dual may be explicitly constructed [18] as a *geometric* orbifold of the IIA string on  $K3$  with a RR Wilson line turned on, but more generally the IIA theory is a nongeometric orbifold of a  $K3$  compactification, as discussed above.

### 3.1. Cyclic Orbifolds of the Leech Lattice CFT

Begin by considering the group of automorphisms of the Leech lattice conformal field theory  $\hat{\Lambda}$ . Cyclic orbifolds of the holomorphic CFT based on the Leech lattice have previously been considered in [28–32]. Closely related constructions based on other  $c = 24$  holomorphic CFTs have been considered in [33,34,35]. We denote the Leech lattice by  $\Lambda$ . As explained more generally above, the automorphism group of the Leech lattice conformal field theory contains the Conway group  $\cdot 0$ , the automorphism group of the Leech lattice [36], but is extended by the automorphism group of the cocycle operators.

A highest weight state in the untwisted Hilbert space is constructed as

$$|\beta\rangle = V(\beta, 0)|0\rangle , \quad (3.5)$$

where  $\beta \in \Lambda$  and the vertex operator  $V(\beta, z)$  is defined as

$$V(\beta, z) =: e^{i\beta \cdot x(z)} : c(\beta) , \quad (3.6)$$

where  $c(\beta)$  is an element in the central extension of  $\Lambda$  by  $\mathbb{Z}_2$ . Associativity of the vertex operator algebra requires that the  $c(\beta)$  satisfy cocycle conditions

$$\begin{aligned} c(\alpha)c(\beta) &= \epsilon(\alpha, \beta)c(\alpha + \beta) \\ \epsilon(\alpha, \beta) &= (-1)^{\alpha \cdot \beta} \epsilon(\beta, \alpha) . \end{aligned} \quad (3.7)$$

For  $\bar{a} \in \cdot 0$ ,  $\bar{a} : \Lambda \rightarrow \Lambda$  and the metric is preserved  $\bar{a}\alpha \cdot \bar{a}\beta = \alpha \cdot \beta$ . The action on the untwisted Hilbert space is simply  $|\beta\rangle \rightarrow |\bar{a}\beta\rangle$ . However, due to the cocycle factors that appear in (3.6) the full automorphism group of  $\hat{\Lambda}$  is the central extension of  $\cdot 0$  by  $\mathbb{Z}_2^{24}$  which is denoted  $C_0 = 2^{24}(\cdot 0)$ . For  $a \in C_0$  the action on the CFT is

$$\begin{aligned} a : c(\beta) &\rightarrow (-1)^{f_a(\beta)} c(\bar{a}\beta) \\ a : |\beta\rangle &\rightarrow (-1)^{f_a(\beta)} |\bar{a}\beta\rangle , \end{aligned} \tag{3.8}$$

where  $\bar{a}$  is the element in  $\cdot 0$  corresponding to  $a$ , and  $f_a(\beta)$  is a  $\mathbb{Z}_2$  central extension of  $\bar{a}$ , with  $f_a(\alpha + \beta) = f_a(\alpha) + f_a(\beta)$ . A useful representation is

$$f_a(\beta) = \beta \cdot m , \tag{3.9}$$

where  $m$  lies on the dual lattice. If  $e_r$  is a basis for the dual lattice, then  $m = m^r e_r$  with  $m^r = 0, 1$ .

Associated to any element  $\bar{a}$  of  $\cdot 0$  of order  $n$  is a characteristic polynomial

$$\det(x - \bar{a}) = \prod_{k|n} (x^k - 1)^{a_k} , \tag{3.10}$$

where

$$\sum_k k a_k = 24 , \tag{3.11}$$

and the  $a_k$  are integers. The symbol  $\prod_{k|n} k^{a_k}$  is known as the Frame shape of the element. Frame shapes for all elements in  $\cdot 0$  are tabulated in [37]. If  $\bar{a}$  lies in the  $M_{24}$  subgroup of  $\cdot 0$ , which has a representation as permutations of 24 letters, then the Frame shape is simply the cycle decomposition of the permutation.

Now we are ready to consider  $\mathbb{Z}_n$  orbifolds by elements in  $C_0$ . We first consider the case when  $m = 0$ . In general, when a symmetry  $a \in C_0$  is lifted to a symmetry of the orbifold the order of  $a$  may increase to  $2n$  when acting on states in the twisted sector. This point is discussed for example in [38]. The result is that  $a$  will be of order  $n$  provided  $n\beta_a^2 = 0 \bmod 2$ , where  $\beta_a$  is the component of  $\beta$  invariant under  $a$

$$\beta_a = 1/n \sum_{k=0}^{n-1} a^k \beta . \tag{3.12}$$

This condition holds for all the automorphisms of the Leech lattice [29] thus provided  $m = 0$ , the action of  $a$  in the twisted sector will always be of order  $n$ .

The vacuum energy of the left-movers is

$$E_L = -1 + \frac{1}{4} \sum_{i=1}^{24} r_i (1 - r_i) , \tag{3.13}$$

where  $\exp(2\pi i r_i)$  are the eigenvalues of coordinates of  $\Lambda$  under the action of  $\bar{a}$  in a diagonal basis. Using (3.11) we find

$$E_L = -\frac{1}{24} \sum_{k|n} \frac{a_k}{k} . \quad (3.14)$$

To obtain a perturbatively consistent theory we need to impose the level-matching condition

$$n(-\frac{1}{24} \sum_{k|n} \frac{a_k}{k} + \frac{1}{2} \delta^2) = 0 \bmod 1 , \quad (3.15)$$

where  $\delta^2 = \delta_L^2 - \delta_R^2$ .

It appears it is possible to construct a shift vector of order  $n$  which satisfies this condition for every element in  $\cdot 0$ . A sample of such orbifolds is listed in the table in appendix A. The first six entries recover all of the cyclic abelian automorphisms appearing in Nikulin's classification of symplectic symmetries of classical  $K3$  surfaces, and were obtained previously as orbifold dual pairs in six and four dimensions [16,18,19]. They satisfy level matching without additional shift vectors, though here we include a shift to eliminate massless states in the twisted sectors. The next four entries are quantum  $K3$  automorphisms. All of the above automorphisms, and the extension to cyclic products of these automorphisms  $\prod_k \mathbb{Z}_{N_k}$  considered by Mukai fit within the finite group  $M_{23}$ .

The next seven entries of the table are quantum  $K3$  symmetries that fit within the finite group  $M_{24}$ , but not  $M_{23}$ . A difference with the symmetries already considered is that now a nontrivial shift vector is required to satisfy level matching. All of these examples contain radial moduli which allow us to decompactify smoothly to six dimensional theories. The remaining entries— including the  $\mathbb{Z}_{20}$  and  $\mathbb{Z}_{60}$  elements contained in the Conway group, but not in  $M_{24}$ — are examples of Leech lattice orbifolds that cannot be decompactified to six dimensions. Note that there is considerable freedom in the choice of the shift vector, giving identical low-energy theories. Under the duality map, this corresponds to the freedom in choosing Ramond-Ramond fluxes in the Type IIA theory. This gives rise to T-duality symmetries of the Type IIA theory which do not originate in conformal field theory!

More generally, one may consider an element  $a$  in  $C_0$  corresponding to nonzero  $m$ . Now one finds that the action of  $a$  on states in the twisted sectors is of order  $2n$  in general [29]. The vacuum energy of the left-movers is modified to

$$E_L = -\frac{1}{24} \sum_{k|n} \frac{a_k}{k} + \frac{1}{8} m_a^2 . \quad (3.16)$$

The level-matching condition now becomes

$$2n(-\frac{1}{24} \sum_{k|n} \frac{a_k}{k} + \frac{1}{8} m_a^2 + \frac{1}{2} \delta^2) = 0 \bmod 1 . \quad (3.17)$$

A simple example of such an orbifold is to take  $\bar{a} = 2^{12}$  and  $m = (1, 0^{23})$ . This is to be combined with a right-moving order 4 shift  $\delta = (2, 1, 1, 1, 1)/4$ .



### 3.2. Monstrous moonshine

Before proceeding to orbifolds of the monster module CFT, let us review the monstrous moonshine conjectures of Conway and Norton [39], which have subsequently been proven by Borcherds [40], and establish some notation and terminology which will be useful later. A helpful review of many of these ideas aimed at string theorists may be found [27]. Consider the fundamental domain of the modular group  $\mathbb{H}/SL(2, \mathbb{Z})$ , where  $\mathbb{H}$  is the upper half plane. Compactifying this space by adjoining the point at infinity, one obtains a space with the topology and complex structure of the Riemann sphere. The modular function  $j$  gives a one-to-one and onto map from  $\mathbb{H}/SL(2, \mathbb{Z}) \cup i\infty$  to the Riemann sphere  $\mathbb{C} \cup \infty$ , thus the Riemann surface  $\mathbb{H}/SL(2, \mathbb{Z}) \cup i\infty$  has genus zero. This means that the field of meromorphic functions on  $\mathbb{H}/SL(2, \mathbb{Z}) \cup i\infty$  is just the rational functions of  $j$  with complex coefficients.  $j$  is said to be the *hauptmodul* of this function field. It is also true that other discrete subgroups  $\Gamma$  of  $SL(2, \mathbb{R})$  yield genus zero Riemann surfaces as the compactification of  $\mathbb{H}/\Gamma$ . Associated with these subgroups are hauptmoduls  $j_\Gamma$  which generate the genus zero function field.

A Thompson series for an element  $g$  of the monster  $\mathbb{M}$  is defined as

$$T_g(q) = q^{-1} \text{Tr} g q^{L_0} , \quad (3.18)$$

where the trace runs over all the states in the monster module  $V^\natural$ . As discussed in [41],  $V^\natural$  may be constructed as a  $\mathbb{Z}_2$  orbifold of the Leech lattice conformal field theory, where the  $\mathbb{Z}_2$  acts by changing the sign of all the left-moving coordinates. Conway and Norton conjectured that the  $T_g$  are hauptmoduls for some genus zero subgroup  $\Gamma_g$  of  $SL(2, \mathbb{R})$ . Properties of these subgroups and hauptmoduls may be found for each conjugacy class of the monster in the tables of [39]. For  $g$  of order  $n$ ,  $T_g$  is fixed by the subgroup  $\Gamma_0(n)$  up to  $h^{\text{th}}$  roots of unity, where  $h|n$  and  $h|24$ . This subgroup is defined by

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{n} \right\} . \quad (3.19)$$

$T_g$  is invariant under  $\Gamma_0(N)$  with  $N = nh$ . The maximal fixing group  $\Gamma_g$  lies between  $\Gamma_0(N)$  and the normalizer of  $\Gamma_0(N)$  in  $SL(2, \mathbb{R})$ . The normalizer of a subgroup  $H$  of  $G$  consists of the elements  $g \in G$  such that  $g^{-1}hg \in H$  for all  $h \in H$ . This normalizer contains the Fricke involution ( $w_N : \tau \rightarrow -1/N\tau$ ). The conjugacy classes with  $h = 1$  will be referred to as *normal*, the others as *anomalous*. The maximal subgroup of  $SL(2, \mathbb{R})$  which fixes  $T_g$  up to an  $h^{\text{th}}$  root of unity will be called the eigenvalue group  $E(g)$ . Also we will refer to classes which contain the Fricke involution in  $E(g)$  as *Fricke* classes.

In the notation of [39],  $E(g)$  is normal when it is of the form  $n + e, f, g, \dots$ . The  $n$  denotes the group  $\Gamma_0(n)$  and the  $e, \dots$  denote Atkin-Lehner involutions  $w_e$  adjoined onto  $\Gamma_0(n)$ .  $E(g)$  is anomalous when it is of the form  $n|h + e, f, \dots$ . The symbol  $n|h$  denotes the group of matrices  $\Gamma_0(n|h)$

$$\begin{pmatrix} a & b/h \\ cn & d \end{pmatrix} , \quad (3.20)$$

with determinant one. For  $\Gamma_0(n|h)$  the Fricke involution is  $w_{n/h}$ . The  $w_e$  (where  $e \in \mathbb{Z}$ ) for  $\Gamma_0(n|h)$  are defined as the set of matrices

$$\begin{pmatrix} ae & b/h \\ cn & de \end{pmatrix} , \quad (3.21)$$

with determinant  $e$ ,  $a, b, c, d \in \mathbb{Z}$  and  $e \parallel n/h$  (i.e.  $e$  divides  $n/h$  and the greatest common divisor  $(e, n/he) = 1$ ).

### 3.3. Cyclic orbifolds of the Monster Module CFT

Now let us consider cyclic orbifolds of the monster module conformal field theory. All the orbifolds we obtain this way will have the maximal rank reduction 24, and will only exist in two spacetime dimensions. The simplest type of elements to consider are the non-Fricke normal elements. There are 38 such conjugacy classes in the monster. The prime order  $p$  classes of this type satisfy  $p - 1 \mid 24$ , and have  $\Gamma_g = \Gamma_0(p)$ . It turns out for each of these 38 classes there is a corresponding element in  $\cdot 0$ , so we may think of these orbifolds as simply automorphisms acting on the Leech lattice, which may then be lifted to  $V^\natural$ . The partition function in the sector twisted by  $g$  is obtained by acting with  $S : \tau \rightarrow -1/\tau$  on  $T_{g^{-1}}$ ,

$$Z(1, g) = q^{-1} \text{Tr}_{H_g} q^{L_0} = T_{g^{-1}}(Sq) . \quad (3.22)$$

Since the element is normal,  $Z(1, g)$  is invariant under  $\tau \rightarrow \tau + n$ , therefore the twisted sector satisfies level matching. We wish to accompany the action of  $g$  by a shift, to ensure that no massless states appear in the twisted sectors. For each of the 38 classes it is possible to construct a purely right-moving shift that acts in the  $\Gamma^{(0,8)}$  component of the lattice, and which satisfies level matching  $n\delta^2/2 = 0 \bmod 1$ .

Now let us consider orbifolds by the 82 normal Fricke classes of the monster. In general, these symmetries do not lie in the  $\cdot 0$  subgroup of the monster. However it is still true that the partition function in the sector twisted by  $g$  is given by a Thompson series as in (3.22), so we will assume the twisted sector  $\mathcal{H}_g$  is well-defined. The general construction of orbifolds based on such nongeometrical symmetries has been studied in detail in [42]. Since the element is normal, the action of  $g$  satisfies level matching. The shifts are constructed as in the preceding paragraph.

Orbifolds generated by the 51 anomalous classes of the monster are somewhat more subtle. The twisted sector does not satisfy level matching unless the action of the symmetry is combined with a nontrivial purely right-moving shift vector  $\delta$ . Consider an element  $g \in \mathbb{M}$  with eigengroup  $n|h + e, f, \dots$ . Under the transformation  $\tau \rightarrow \tau + n$ , the partition function for the twisted sector without a shift picks up a phase  $e^{2\pi i l/h}$ , where  $l$  is some integer. The global phase anomaly is canceled by combining the action of  $g$  with a shift  $\delta$  which satisfies  $n(\delta^2/2 + l/h) = 0 \bmod 1$ . Such shifts may be constructed for each element of the monster, however the analysis is performed case by case.

We present some examples of these orbifolds in the table in appendix B.

## 4. New N=2 Type IIA/Heterotic Dual Pairs

It is straightforward to obtain dual pairs with  $N = 2$  supersymmetry in 4d from the subclass of maximally supersymmetric dual pairs which may be decompactified to six dimensions. Within this class, before orbifolding to obtain the reduced  $N = 4$  theory, there is a region in moduli space on the heterotic side in which the Narain lattice decomposes as

$$\Gamma^{(24,8)} = \Gamma^{(20,4)} \oplus \Gamma^{(4,4)} . \quad (4.1)$$

On the Type II side, where we start with a compactification on  $K3 \times T^2$ , we may orbifold by the  $(-1)^{F_L}$  symmetry combined with a  $\mathbb{Z}_2$  shift in the  $T^2$  satisfying  $\delta^2 = 0$ , along the lines of the constructions in [17,15]. As discussed in these references,  $(-1)^{F_L}$  acts as a change in sign of the  $\Gamma^{(20,4)}$  component of the lattice on the heterotic side. To satisfy level matching an additional shift in the  $\Gamma^{(4,4)}$  component of the lattice is required, satisfying  $\tilde{\delta}^2 = 1/2$ . On each side, these  $\mathbb{Z}_2$  orbifolds break the supersymmetry down to  $N = 2$ . One may then proceed with the construction of reduced theories as described in the previous sections, orbifolding by some discrete group  $G$ . In general, one typically needs to also check level matching is satisfied in the sectors twisted by the product of  $\mathbb{Z}_2$  with an element in  $G$ . In these theories one is left with a low energy spectrum which contains four  $U(1)$  gauge fields at a generic point and  $20 - \Delta r$  hypermultiplets, where  $\Delta r$  is the gauge group rank reduction of the corresponding  $N = 4$  theory.

The resulting Type II theories will have  $(1, 4)$  worldsheet supersymmetry. As discussed in [15] the dilaton in such theories lies in a vector multiplet. The dilaton on the heterotic side likewise lies in a vector multiplet. In each case we therefore expect quantum corrections to the vector multiplet moduli space. This class of dual pairs does not display second-quantized mirror symmetry [7]. The hypermultiplet moduli space does not receive quantum corrections on either side. Unlike the examples considered in [6], it is possible to obtain hypermultiplet moduli spaces of low dimension. We plan to return to these models in future work.

We emphasize that the four dimensional  $N=2$  Type IIA vacua we are considering do *not* have a straightforward conformal field theory description since they involve Ramond-Ramond backgrounds. Thus, they are also the simplest examples of the new Type IIA vacua proposed in [43]. The backgrounds are described by electromagnetic flux configurations, localized at fixed points on the compactification.

## 5. Conclusions

Our analysis has been restricted to the simplest Type IIA vacua which lie outside of conformal field theory, and to phenomena that occur in the generic moduli space of these theories. More generally, Type II string theories may contain a variety of both soft [44,45,46] and hard [47,48] BPS saturated p-brane solitons, including those with RR charge. In a beautiful recent insight [48], it has been shown that the quantum degrees of freedom required by string duality are the Dirichlet-branes of a Type I string theory [47]. D-branes potentially provide a powerful calculus for the dynamics of BPS states in string theory. They should provide a much more explicit description of nonperturbative string effects, such as nonabelian gauge symmetry, associated with regions of the moduli space where the Type IIA theory is strongly coupled [49,12,8,50,10].

An essential ingredient underlying our work was the resolution in [16] of the paradox posed by the existence of a six dimensional  $N=2$  heterotic model with eight fewer abelian gauge fields at generic points in the moduli space [26,51] than the usual model obtained via toroidal compactification. The Type IIA dual cannot correspond to a conformal field theory compactification [13]. It is described instead by compactification on a  $K3/\sigma$  orbifold with a Ramond-Ramond  $\mathbb{Z}_2$  flux localized at fixed points of the involution  $\sigma : U^i \rightarrow -U^i$ ,  $i = 1,$

$\dots$ , 8 for eight of the harmonic forms on the  $K3$  surface [16]. We have shown the necessity of extending the class of Ramond-Ramond electromagnetic flux configurations considered on the Type II side to accommodate the generic strong/weak Type IIA/heterotic orbifold dual pair in six dimensions and below. We suspect this is only the tip of the iceberg, and that much remains to be discovered about generic Ramond-Ramond backgrounds and their relationship to Type IIA/heterotic duality.

In particular, consider the 19 anti-self dual (ASD) harmonic forms on the  $K3$  surface. There is a one-to-one correspondence between the ASD forms, defining the left moving part of the integral cohomology lattice, and Ramond-Ramond gauge fields which arise from integrating the RR three-form potential over the associated non-trivial two-cycles of the  $K3$ . Consider a compactification  $K3/G$ , where we introduce Ramond-Ramond  $\mathbb{Z}_2$  gauge transformations associated to directions of the lattice for which  $G$  has some symplectic action. Such an action is mapped under duality to an automorphism of the heterotic conformal field theory which introduces *nonabelian* discrete torsion, reminiscent of the fermionic construction of [26].<sup>4</sup>

To see the relation to the fermionic models of [26] recall that the automorphism group of a conformal field theory is larger than that of the underlying lattice. For an  $r$  dimensional lattice, this extension is the  $(\mathbb{Z}_2)^r$  automorphism group of the cocycles required by associativity of the operator product algebra [27]. Automorphisms of the lattice which only satisfy level matching when accompanied by nontrivial action on the cocycles are associated with twisted vertex operator algebras [52,38]. These are straightforwardly realized on the world-sheet by Majorana-Weyl fermions [26]. This resolves the puzzle that not all of the fermionic models of [26] were obtained as orbifolds with respect to abelian lattice automorphisms plus shifts in an invariant direction [51,18].

In this paper we have explored an algebraic structure contained in the monster group underlying Type II/heterotic duality. Recent work has uncovered other connections to the monster. As discovered by Harvey and Moore [53], the monster Lie superalgebra miraculously appears in the algebra of the tower of elementary BPS string states of a four dimensional  $N=2$  heterotic string theory. They have further speculated that different dual theories are simply alternate representations of an underlying algebraic structure, much like the different realizations of an affine Lie algebra. A related observation has been made by Lian and Yau who have found that Thompson series are related to  $K3$  mirror maps [54]. Further elucidating these connections will no doubt lead to a better understanding of strong/weak duality in string theory.

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<sup>4</sup> A similar phenomenon occurs in the Type I theory [8]. We would like to thank J. Polchinski for pointing out this connection.

## Appendix A. Examples of cyclic orbifolds of the Leech lattice CFT

Column 1 gives the order of the automorphism. The second column gives the Frame shape of the element in the Conway group. If the element lies in the  $M_{24}$  subgroup of  $\cdot 0$ , then the Frame shape is simply the cycle decomposition of the permutation. The third column gives an example of a purely right-moving shift vector required for level matching. All of the examples, except for the last, contain radial moduli. By decompactifying with respect to these moduli higher dimensional theories are obtained. The fourth column gives the maximum number of noncompact dimensions that may be obtained this way, such that the theory still possesses a Type IIA dual. Finally, the fifth column gives the overall gauge multiplet rank reduction.

$\mathbb{Z}_n$	$g$	$\delta_R$	$d_{max}$	$-\Delta r$
$\mathbb{Z}_2$	$1^8 2^8$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	6	8
$\mathbb{Z}_3$	$1^6 3^6$	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	6	12
$\mathbb{Z}_5$	$1^4 5^4$	$(\frac{3}{5}, \frac{1}{5}, 0, 0)$	6	16
$\mathbb{Z}_6$	$1^2 2^2 3^2 6^2$	$(\frac{3}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$	6	16
$\mathbb{Z}_7$	$1^3 7^3$	$(\frac{3}{7}, \frac{2}{7}, \frac{1}{7}, 0)$	6	18
$\mathbb{Z}_8$	$1^2 \cdot 2 \cdot 4 \cdot 8^2$	$(\frac{3}{8}, \frac{2}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$	5	18
$\mathbb{Z}_{11}$	$1^2 11^2$	$(\frac{4}{11}, \frac{2}{11}, \frac{1}{11}, \frac{1}{11})$	6	20
$\mathbb{Z}_{14}$	$1 \cdot 2 \cdot 7 \cdot 14$	$(\frac{5}{14}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14})$	6	20
$\mathbb{Z}_{15}$	$1 \cdot 3 \cdot 5 \cdot 15$	$(\frac{5}{15}, \frac{2}{15}, \frac{1}{15}, 0)$	6	20
$\mathbb{Z}_{23}$	$1 \cdot 23$	$(\frac{6}{23}, \frac{3}{23}, \frac{1}{23}, 0)$	4	22
$\mathbb{Z}_2$	$2^{12}$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	6	12
$\mathbb{Z}_3$	$3^8$	$(\frac{2}{3}, 0, 0, 0)$	6	16
$\mathbb{Z}_4$	$2^4 4^4$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	6	16
$\mathbb{Z}_4$	$4^6$	$(\frac{2}{4}, \frac{1}{4}, \frac{1}{4}, 0)$	6	18
$\mathbb{Z}_6$	$6^4$	$(\frac{2}{6}, \frac{2}{6}, \frac{1}{6}, \frac{1}{6})$	6	20
$\mathbb{Z}_{10}$	$2^2 10^2$	$(\frac{3}{10}, \frac{1}{10}, 0, 0)$	6	20
$\mathbb{Z}_{12}$	$2 \cdot 4 \cdot 6 \cdot 12$	$(\frac{3}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$	6	20
$\mathbb{Z}_4$	$1^4 2^2 4^4$	$(\frac{2}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	5	14
$\mathbb{Z}_{12}$	$12^2$	$(\frac{4}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12})$	4	22
$\mathbb{Z}_{21}$	$3 \cdot 21$	$(\frac{5}{21}, \frac{1}{21}, \frac{1}{21}, \frac{1}{21})$	4	22
$\mathbb{Z}_{20}$	$4 \cdot 20$	$(\frac{5}{20}, \frac{2}{20}, \frac{1}{20}, 0)$	4	22
$\mathbb{Z}_{60}$	$3 \cdot 4 \cdot 5 \cdot 60 / 1 \cdot 12 \cdot 15 \cdot 20$	$(\frac{10}{60}, \frac{4}{60}, \frac{1}{60}, \frac{1}{60}, \frac{1}{60}, \frac{1}{60})$	2	24

## Appendix B. Examples of cyclic orbifolds of the monster module CFT

The first column is the conjugacy class of the element in  $g \in \mathbb{M}$ . The second column gives the eigengroup  $E(g)$ , and the third column gives the shift vector.

$g$	$E(g)$	$\delta_R$
2A	2+	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
2B	2-	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
3A	3+	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$
3B	3-	$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$
4D	4 2-	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
5A	5+	$(\frac{3}{5}, \frac{1}{5}, 0, 0)$
5B	5-	$(\frac{3}{5}, \frac{1}{5}, 0, 0)$
7A	7+	$(\frac{3}{7}, \frac{2}{7}, \frac{1}{7}, 0)$
7B	7-	$(\frac{3}{7}, \frac{2}{7}, \frac{1}{7}, 0)$
11A	11+	$(\frac{4}{11}, \frac{2}{11}, \frac{1}{11}, \frac{1}{11})$
13B	13-	$(\frac{5}{13}, \frac{1}{13}, 0, 0)$

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